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COMPLEMENTED IDEALS IN THE DISK ALGEBRA

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ABSTRACT

The complemented ideals in the disk algebra are characterized. The projection operators onto the ideals and the complements of the ideals are identified.

1. Let A denote the disk algebra. This is the set of analytic functions on the open unit disk in the complex plane which are continuous on the closed unit disk. It is a Banach space under the supremum norm. If K is a compact subset of the unit circle with Lebesgue measure zero, let A_K denote the ideal in A consisting of functions which vanish on K. The most general closed ideals in A have the form

$$J_F = \{g \cdot F : g \in A_K\} \tag{1}$$

where F is an inner function continuous on the complement of K in the closed disk.†† When the set K is predetermined we will say that J_F is the ideal generated by F.

A sequence of points $\{z_n\}$ in the unit disk will be called a *Carleson sequence* if the measure which assigns, for each n, a mass of $1-|z_n|^2$ to the point z_n (taking multiplicities into account) is a Carleson measure. That is, if there is a constant C so that whenever $0<h<1$ and x real,

$$\sum_{z_n \in S_h(x)} (1-|z_n|^2) \leqq Ch, \tag{2}$$

where

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†† A convenient reference for the basic properties of the disk algebra is [7].

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(3) $$S_h(x)=\left\{re^{i\theta} : i-h<r<1 \text{ and } |x-\theta|<\frac{h}{2}\right\}.$$

The main result of this paper can be expressed as

THEOREM 1. *A closed ideal in the disk algebra is a complemented subspace if and only if it is generated by a Blaschke product whose zeros form a Carleson sequence.*

The proof that ideals of the form described are complemented is given in Section 2. The projections with such ideals as kernels are explicitly described and their ranges are determined. That is, the subspaces complementary to the ideals are identified. In Section 3 it is shown that the only complemented ideals are those described in Theorem 1. In Section 4 some remarks are made concerning the norms of the projections onto ideals. Finally, we remark that similar arguments can be used to classify the complemented ideals in H^∞ of the form FH^∞, where F is an arbitrary inner function.

2. Throughout the paper, let K denote a fixed compact subset of the unit circle with measure zero. The restriction of elements in A to K defines a linear operator $R: A \to C(K)$ of norm one whose kernel is the ideal A_K. The Rudin–Carleson theorem asserts that this operator is onto $C(K)$. In [9], Michael and Pelczynski construct a linear extension $E: C(K) \to A$ of norm one which provides a right inverse for R. It follows that $E \circ R$ is a projection on A whose kernel is A_K and whose range is isomorphic (as a Banach space) to $C(K)$. Thus, in proving Theorem 1 it suffices to consider when ideals of the form (1) are complemented in A_K.

Given a sequence of points $\{z_n\}$ in the unit disk we can define another restriction operator T by assigning to each f in A_K the sequence $Tf = \{f(z_n)\}$. If $\{z_n\}$ is an infinite sequence whose accumulation points all lie in K, then $T: A_K \to c_0$ (the space of null sequences) is a linear operator of norm one. The kernel of T is the collection of functions in A_K which vanish at each z_n. If the points $\{z_n\}$ are distinct and $\Sigma 1-|z_n|<\infty$, then the kernel of T is the ideal J_B, where B is the Blaschke product with zeros $\{z_n\}$.

The sequence $\{z_n\}$ is called an *interpolation sequence* if T is onto c_0. In this case $\{z_n\}$ is also a Carleson sequence (see [2], [4], or [7]). Moreover, a bounded linear interpolation operator $I: c_0 \to A_k$, which is a right inverse for T, can be constructed. In [3] (see also [5], [6]), Carleson proves the following result, due to Beurling:

THEOREM 2. *If $\{z_n\}$ is an interpolation sequence all of whose accumulation points lie in K, then there is a sequence $\{f_n\}$ in A_K so that*

(4) $$f_n(z_m) = \delta_{nm} \quad \text{and} \quad \sup_{|z|<1} \sum |f_n(z)| < \infty.$$

Using these functions, we can define the operator I by

$$I(\{w_n\}) = \sum_n w_n f_n, \qquad \text{for } \{w_n\} \in c_0.$$

Therefore, for $g \in A_k$,

(5) $$(I \circ T)g = \sum_n g(z_n) f_n$$

defines a projection on A_K whose kernel is J_B and whose range is isomorphic to c_0.

We remark that any projection on A_K with J_B as its kernel has the form (5) where $\{f_n\}$ satisfies the conditions (4).

The step from interpolation sequences to Carleson sequences is made by way of

PROPOSITION 3. *Let F_1 and F_2 be inner functions which are continuous off K. Put $F = F_1F_2$ and let J, J_1 and J_2 be the ideals in A_K generated by F, F_1 and F_2 respectively. If P_1 and P_2 are projections on A_K whose kernels are J_1 and J_2 respectively, then there exists a projection P on A_K, whose kernel is J and whose range is isomorphic to the direct sum of the ranges of P_1 and P_2.*

PROOF. For $g \in A_K$, define

(6) $$Pg = F_1 \cdot P_2(\bar{F}_1 \cdot (I - P_1)g) + P_1 g.$$

A computation shows that P is a projection on A_K. Since A_K is the direct sum of J_1 and the range of P_1, g lies in the kernel of P if and only if both $P_1 g = 0$ and $P_2(\bar{F}_1(I - P_1)g) = 0$. It follows that J is the kernel of P. The range of P is the direct sum of the ranges of P_1 and of $F_1 \cdot P_2(\bar{F}_1(I - P_1))$. Since multiplication by F_1 is an isometry on A_K, the range of $F_1P_2(\bar{F}_1(I - P_1))$ is isomorphic to the range of $P_2(\bar{F}_1(I - P_1))$. However, this is the range of P_2 since $\bar{F}_1(I - P_1)$ maps A_K onto A_K, and the proposition follows.

The projections onto ideals of the form (1) can be realized in the following way. The function from J_F to A_K defined by "dividing by F" is a linear map of norm one. If this map has a bounded linear extension $D : A_K \to A_K$, then

$Pf = F \cdot Df$ is a projection from A_K onto J_F. Conversely, given such a projection, $Df = \bar{F} \cdot Pf$ defines a bounded linear extension of the division map.

The projection (6) is the result of successive divisions by F_1 and F_2. In special cases, the projection P in (6) can be computed explicitly. For example, if $F_1 = F_2$ is a Blaschke product whose zeroes form an interpolation sequence and P_1 has the form (5), then all the zeroes of F have multiplicity two and Pg can be written as a series involving the values of g and the values of the derivative of g at the sequence of zeroes.

An immediate consequence of Proposition 3 is that an ideal J_B is complemented whenever B can be written as a finite product of Blaschke products each of whose zeros form an interpolation sequence. Since any Carleson sequence can be expressed as a finite union (taking into account multiplicities) of interpolation sequences (see [8]), the sufficiency of the conditions given in Theorem 1 is established. More precisely, since finite direct sums of c_0 with itself are isomorphic to c_0, we have proven

THEOREM 4. *If B is a Blaschke product whose zeros form a Carleson sequence and which accumulate only on K, then J_B is a complemented subspace of A_K whose complement is isomorphic to c_0.*

3. In general, an inner function F can be factored into a Blaschke product and a singular inner function. To complete Theorem 1, we need to show if an ideal J_F is complemented then the singular factor is constant and the zeros of the Blaschke product form a Carleson sequence. This will be accomplished in the following Propositions.

PROPOSITION 5. *Using the notation of Proposition 3, if there is a projection Q from A_K onto J, then there are projections Q_1 and Q_2 from A_K onto J_1 and J_2 respectively so that $\|Q_j\| \leqq \|Q\|$, for $j = 1, 2$.*

PROOF. For $g \in A_K$, define $Q_1 g = \bar{F}_2 \cdot Q(F_2 g)$ and $Q_2 g = \bar{F}_1 \cdot Q(F_1 g)$.

PROPOSITION 6. *There is a constant $M > 0$ so that whenever F is an inner function with a zero of multiplicity n and Q is a projection from A_K onto J_F, then $\|Q\| \geqq M \log n$.*

PROOF. If ϕ is a conformal map of the disk onto itself, then composition with ϕ transforms A_K into $A_{\phi^{-1}(K)}$ and J_F into $J_{F \circ \phi}$. Also, there will be a projection from $A_{\phi^{-1}(K)}$ onto $J_{F \circ \phi}$ with the same norm as Q. Thus, without loss of generality, we can assume F has a zero of multiplicity n at the origin. Then F is divisible by z^n so, by Proposition 5, ther is a projection Q_1 onto J_{z^n} with $\|Q_1\| \leqq \|Q\|$. Since the projection of smallest norm onto J_{z^n} is the Fourier projection (see [7]), the result follows.

PROPOSITION 7. *Let F be an inner function continuous off K and Q a projection from A_K onto J_F. Let $0 < \varepsilon < 1/\|Q\|$. Whenever G is an inner function continuous off K so that $\|F - G\|_\infty < \varepsilon$, there exists a projection Q_1 from A_K onto J_G with*

$$\|Q_1\| \leqq \frac{\|Q\|}{1 - \varepsilon \|Q\|}.$$

PROOF. The mapping $f \to f \cdot \bar{F} \cdot G$ is an isometry of J_F onto J_G satisfying,

$$\|f - f \cdot \bar{F} \cdot G\| \leqq \|f\| \|F - G\| \leqq \|f\| \varepsilon.$$

A standard perturbation argument (see for example [1], lemma 1) shows that if J_F is complemented so is J_G and the stated inequality holds.

PROPOSITION 8. *If J_F is complemented in A_K, then F is a Blaschke product.*

PROOF. Let S be the singular factor of F. By Proposition 5, if J_F is complemented so is J_S. Let Q be a projection from A_K onto J_S. With M as in Proposition 6, choose n so that $M \log n > 2\|Q\|$. Let $\varepsilon > 0$ be chosen so that $\varepsilon \|Q\| < \frac{1}{2}$. Since S has no zeros, it has an analytic nth root $S^{1/n}$ which is also an inner function. There is a Blaschke product B, continuous off K, so that $\|S^{1/n} - B\|_\infty < \varepsilon / n$ and hence so that $\|S - B^n\|_\infty < \varepsilon$. Proposition 7 now implies that there is a projection onto J_{B^n} of norm at most $2\|Q\|$. If B had any zeros, we would have a contradiction to Proposition 6. Thus B, and hence S, must be a constant and the proof is complete.

PROPOSITION 9. *Let B be a Blaschke product whose zeros do not form a Carleson sequence. Let $\varepsilon > 0$ and n be a positive integer. Then, there are Blaschke products $B_1, B_2, \cdots, B_n$ and a point z_0 in the disk so that $B = B_1 \cdot B_2 \cdots B_n$ and $|B_j(z_0)| < \varepsilon$ for $1 \leqq j \leqq n$.*

PROOF. If $0 < h < 1/3$ and x is real, put $z_0 = (1 - 3h)e^{ix}$. Let $S_h(x)$ be as in (3). A geometric argument then shows that

$$\left| \frac{z - z_0}{1 - \bar{z} z_0} \right| \geqq \frac{1}{2}, \qquad \text{for } z \in S_h(x). \tag{7}$$

Using the inequality

$$\log t \geqq \frac{\log 2}{3}(t^2 - 1),$$

valid for $1 \leqq t \leqq 2$, we have, for $z \in S_h(x)$,

$$\log\left|\frac{1-\bar{z}z_0}{z-z_0}\right| \geqq \frac{\log 2}{3}\left(\frac{(1-|z|^2)(1-|z_0|^2)}{|z-z_0|^2}\right) \geqq \frac{\log 2}{16}\,\frac{1-|z|^2}{n}.$$

It follows that if A is a Blaschke product whose zeros $\{w_k\}$ lie in $S_h(x)$ and satisfy $\Sigma\, 1-|w_k|^2 \geqq Ch$ then

$$|A(z_0)| \leqq \exp\left(-\frac{\log 2}{16}C\right). \tag{8}$$

Let the zeros of B be $\{z_k\}$. Since $\{z_k\}$ is not Carleson, for any C there is a set $S_h(x)$ so that

$$\sum_{z_n \in S_h(x)} (1-|z_n|^2) \geqq Ch.$$

If A is the Blaschke product whose zeroes w_k are those of B which lie in $S_h(x)$, then by choosing C sufficiently large, (7) and (8) give

$$\left|\frac{w_k - z_0}{1-\bar{w}_k z_0}\right| \geqq \frac{1}{2} \qquad \text{and} \qquad |A(z_0)| < \left(\frac{\varepsilon}{2}\right)^n.$$

Therefore, there is an l_1 so that

$$\frac{\varepsilon}{2} \leqq \prod_{k=1}^{l_1}\left|\frac{w_k - z_0}{1-\bar{w}_k z_0}\right| < \varepsilon.$$

Put

$$B_1(z) = \prod_{k=1}^{l_1}\frac{w_k - z}{1-\bar{w}_k z}.$$

Then

$$\left|\frac{A(z_0)}{B_1(z_0)}\right| = \prod_{k=l_1+1}^{\infty}\left|\frac{w_k - z_0}{1-\bar{w}_k z_0}\right| < \left(\frac{\varepsilon}{2}\right)^{n-1}.$$

Thus, there is an l_2 so

$$\frac{\varepsilon}{2} \leqq \prod_{k=l_1+1}^{l_2}\left|\frac{w_k - z_0}{1-\bar{w}_k z_0}\right| < \varepsilon.$$

Put

$$B_2(z) = \prod_{k=l_1+1}^{l_2}\frac{w_k - z}{1-\bar{w}_k z}.$$

Continuing, we obtain $B_1, B_2, \cdots, B_{n-1}$ so that $|B_j(z_0)| < \varepsilon$ and so that

$$\left|\frac{A(z_0)}{B_1(z_0)\cdot B_2(z_0)\cdots B_{n-1}(z_0)}\right|<\frac{\varepsilon}{2}.$$

Since A divides B, we can put $B_n = B/B_1\cdot B_2\cdots B_{n-1}$. Then,

$$|B_n(z_0)=\left|\frac{B(z_0)}{B_1(z_0)\cdots B_{n-1}(z_0)}\right|\leqq\left|\frac{A(z_0)}{B_1(z_0)\cdots B_{n-1}(z_0)}\right|<\varepsilon.$$

This completes the proof of the proposition.

We can now complete the proof of Theorem 1. By Proposition 8, if an ideal J_F is complemented in A_K then F is a Blaschke product. Suppose the zeros of B do not form a Carleson sequence. Let Q be a projection from A_K onto J_F. Choose $\varepsilon>0$ so that $\varepsilon\|Q\|<\frac{1}{2}$ and an n so that $M\log n>2\|Q\|$, where M is as in Proposition 6. By Proposition 9, for any $\delta>0$, there is a point z_0 in the disk and a factorization of F into n Blaschke products $B_1,\cdots,B_n$ so that $|B_j(z_0)|<\delta$ for $1\leqq j\leqq n$. Then,

$$G(z)=\prod_{j=1}^{n}\frac{B_j(z)-B_j(z_0)}{1-\overline{B_j(z_0)}B_j(z)}$$

is an inner function, continuous off K, which has a zero of multiplicity at least n at z_0. Moreover, if δ is sufficiently small, then $\|F-G\|_\infty<\varepsilon$. Proposition 7 yields a projection onto J_G of norm at most $2\|Q\|$. This contradicts Proposition 6 and completes the proof.

We will conclude with some remarks concerning the norms of the projections constructed in Section 2. As mentioned in the proof of Proposition 6, the projections of minimal norm on an ideal generated by a Blaschke product with a single multiple zero can be constructed from the Fourier projection. However, even in the case of finitely many distinct zeros, minimal projections, or even their norms, are not explicitly known.

While the supremum in (4) provides an upper bound for the norm of the projection (5), the following example is instructive. In [3], Carleson shows that the supremum in (4) is of the order of $(1/\delta)\log(1/\delta)$, where δ is the interpolation constant of the sequence $\{z_n\}$. That is,

$$\delta=\inf\prod_{j\neq n}\left|\frac{z_j-z_n}{1-\bar{z}_jz_n}\right|.$$

If we let $F=B_1\cdot B_2$ where B_1 and B_2 are finite Blaschke products with distinct zeros but the same interpolation constant δ, then the remarks above and the proof of Proposition 3 show that there are projections onto J_F with norms bounded by expressions depending only on δ. However, by allowing the zeros of

B_1 and B_2 to coalesce we can make the interpolation constant of the zeros of F as small as we please, and so the supremum in (4) as large as we please, while holding δ constant.

We have, however, arrived at the following.

CONJECTURE. *Let $\{z_n\}$ be a Carleson sequence and B, the Blaschke product with zeros $\{z_n\}$, continuous off K. There are absolute constants a and A so that*

(9) $$a \log M \leqq \inf\{\|Q\| : Q : A_k \to J_B \text{ is a projection onto}\} \leqq A \log M$$

where $M = \sup \Sigma_n (1 - |z_n|^2)|f(z_n)| : f \in H^1$ *and* $\|f\|_1 \leqq 1\}$.†

We remark that in case B has a single zero of multiplicity n, (9) follows from Proposition 6.

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† The finiteness of M is equivalent to $\{z_n\}$ being a Carleson sequence, see [4].